Groups in NTP2

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ABSTRACT. We prove the existence of abelian, solvable and nilpotent definable envelopes for groups definable in models of an NTP_2 theory.

1. Introduction

One of the main concerns of model theory is the study of definable sets. For example, given an abelian subgroup in some definable group, whether or not one can find a *definable* abelian group containing the given subgroup becomes very important, since it "brings" objects outside the scope of model theory into the category of definable sets.

In that sense, an ongoing line of research consists of finding "definable envelopes". Specifically, one can ask if for a definable group G and a given subgroup of G with a particular algebraic property such as being abelian, solvable, or nilpotent, can one find a *definable* subgroup of G which contains the given subgroup and which has the same algebraic property. This is always possible in stable theories (see [8]), and recent research has shown remarkable progress both simple theories, and dependent theories:

In a dependent theory, Shelah has shown that given any definable group G and any abelian subgroup of G, one can find a definable abelian subgroup in some extension of G which contains the given abelian subgroup (see [10]). De Aldama generalized this result in [1] to nilpotent and normal solvable subgroups.

In simple theories, one cannot expect such a result to hold, as there are examples of definable groups with simple theories which contain infinite abelian subgroups but for which all its definable abelian subgroups are finite (see Remark 2.7). Nevertheless, one obtains definable envelopes "up to finite index". In [5] and [4] Milliet proved that given any (abelian/nilpotent/solvable) subgroup H of a group G definable in a simple theory one can find a subgroup of G which contains H *up to finite index* and which is (abelian/nilpotent/solvable).[1]

In this paper, we analyze arbitrary abelian, nilpotent and normal solvable subgroups of groups definable in theories without the tree property of the

[1]The existence of nilpotent envelopes played an essential role in the proof of Palacin and Wagner showing that the "fitting subgroup", i. e. the group generated by all normal nilpotent subgroups, of a group definable in a simple theory is again nilpotent (see [7]).

second kind (NTP_2 theories), which include both simple and dependent theories. We prove the existence of definable envelopes up to finite index in a saturated enough extension of a given group which is definable in a model of an NTP_2 theory, which is inspired by the result in simple theories as well as the one in dependent theories.

2. Preliminaries

In this section we state the known results in simple and dependent theories. Throughout the paper, we say that a group is definable in a theory if the group is definable in some model of the theory. We also sometimes say that a group is *dependent*, *simple* or *NTP_2* if the theory of the group, in the language of groups, is, respectively, dependent, simple, or NTP_2. For some cardinal κ, a κ-saturated extension of a definable group is this group "seen" in an κ-saturated extension of the model in which the group is defined.

Definition 2.1. *Let M be a model of a theory T in a language $\mathcal{L}$. Let A be a subset of M. A sequence $\langle a_i \rangle_{i\in I}$ is defined to be indiscernible over A, if I is an ordered index set and given any formula $\phi(x_1, \ldots, x_n)$ with parameters in A, and any two subsets $i_1 < i_2 < \ldots i_n$ and $j_1 < j_2 < \ldots j_n$ of I, we have*

$$M \models \phi(a_{i_1}, \ldots, a_{i_n}) \Leftrightarrow \phi(a_{j_1}, \ldots, a_{j_n}).$$

The following is a well known fact which is proved using Erdös-Rado Theorem.

Fact 2.2. *For some cardinal κ and any set A, any sequence of elements $\langle a_i \rangle_{i\in\kappa}$ contains a subsequence of size ω which is indiscernible over A.*

Even more, for any cardinal λ and any set A, there is some cardinal κ such that any sequence of elements $\langle a_i \rangle_{i\in\kappa}$ contains a subsequence of size λ which is indiscernible over A.

Definition 2.3. *A theory T is* dependent *if in no model M of T one can find an indiscernible sequence $\langle \bar{a}_i \rangle_{i\in\omega}$ and a formula $\phi(\bar{x}; \bar{b})$ such that $\phi(\bar{a}_i; \bar{b})$ holds in M if and only if i is odd.*

Let G be a group definable in a dependent theory, let H be a subgroup of G and let $\mathcal{G}$ be a $|H|^+$-saturated extension of G. The following two results summarize what we know about envelopes of H. The first was proven by Shelah in [10] and the second by de Aldama in [1].

Fact 2.4. *If H is abelian, then there exists a definable abelian subgroup of $\mathcal{G}$ which contains H.*

Fact 2.5. *If H is a nilpotent (respectively normal solvable) subgroup of G of class n, then there exists a definable nilpotent (respectively normal solvable) subgroup of $\mathcal{G}$ of class n which contains H.*

We now turn to the simple theory context.

Definition 2.6. *A theory has the* tree property *if there exists a formula* $\phi(\bar{x};\bar{y})$*, a parameter set* $\{\bar{a}_\mu : \mu \in \omega^{<\omega}\}$ *and* $k \in \omega$ *such that*

- $\{\phi(\bar{x};\bar{a}_{\mu^\frown i} : i < \omega\}$ *is* k*-inconsistent for any* $\mu \in \omega^{<\omega}$*;*
- $\{\phi(\bar{x};\bar{a}_{s\restriction n} : s \in \omega^\omega, n \in \omega\}$ *is consistent.*

A theory is called simple *if it does not have the tree property.*

As the following remark (an example which is studied in [4]) shows, it is impossible to get envelopes in the same way one could achieve them in the stable and dependent case, and one must allow for some "finite noise".

Remark 2.7. *Let* T *be the theory of an infinite vector space over a finite field together with a skew symmetric bilinear form. Then* T *is simple, and in any model of* T *one can define an infinite "extraspecial* p*-group"* G*, i. e. every element of* G *has order* p*, the center of* G *is cyclic of order* p *and is equal to the derived group of* G*. This group has SU-rank 1. It has infinite abelian subgroups but no abelian subgroup of finite index, as the center is finite and any centralizer has finite index in* G*. However, if* G *had an infinite definable abelian subgroup, that abelian group would have SU-rank 1, hence would be of finite index in* G*, a contradiction.*

A model theoretic study of extra special p*-groups can be found in* [2].

So one has to find a version of the theorem which is adapted to the new context. For this we will need the following definitions:

Definition 2.8. *A group* G *is called* finite-by-abelian *if there exists a finite normal subgroup* F *of* G *such that* G/F *is abelian.*

Definition 2.9. *A subgroup* H *of a group* G *is an* almost abelian group *if the centralizer of any of its elements has finite index in* H*. If the index of these elements can be bounded by some natural number we call it an* bounded almost abelian group.

Almost abelian groups are also known as FC-groups, where FC-group stands for "finite conjugation"-group.

The following classical group theoretical result, which is a theorem of Neumann, will provide a link between the two notions.

Fact 2.10. [6, Theorem 3.1]. *Let* G *be a bounded almost abelian group. Then its derived group is finite. In particular,* G *is finite-by-abelian.*

Now we are ready to state the abelian version for simple theories proven by Milliet as [5, Proposition 5.6.].

Fact 2.11. *Let* G *be a group definable in a simple theory and let* H *be an abelian subgroup of* G*. Then there exists a definable finite-by-abelian subgroup of* G *which contains* H*.*

In the nilpotent and solvable case one must additionally include other definitions to account for the "by finite" phenomenon.

Definition 2.12. *Let G be a group and H and K be two subgroups of G. We say that H is* almost contained *in K, denoted by $H \lesssim K$, if $[H : H \cap K]$ is finite.*

The following was proved by Milliet in [4]:

Fact 2.13. *Let G be a group definable in a simple theory and let H be a nilpotent (respectively solvable) subgroup of G of class n. Then one can find a definable nilpotent (respectively solvable) subgroup of class at most $2n$ which almost contains H.*

If we additionally assume that the nilpotent subgroup H is normal in G, one can ask for the definable subgroup which almost contains H to be normal in G as well. Hence the product of these two groups is a definable normal nilpotent subgroup of G of class at most $3n$ which **contains** H.

3. Main result

The purpose of this paper is to extend the above results to the context of NTP_2 theories, which expand both simple and dependent theories.

Definition 3.1. *A theory has the* ***tree property of the second kind*** *(refered to as TP_2) if there exists a formula $\psi(\bar{x};\bar{y})$, an array of parameters $(\bar{a}_{i,j} : i, j \in \omega)$, and $k \in \omega$ such that:*

- *$\{\psi(\bar{x};\bar{a}_{i,j}) : j \in \omega\}$ is k-inconsistent for every $i \in \omega$;*
- *$\{\psi(\bar{x};\bar{a}_{i,f(i)}) : i \in \omega\}$ is consistent for every $f : \omega \to \omega$.*

A theory is called ***NTP_2*** *if it does not have the TP_2.*

Observation. *By compactness, having the tree property of the second kind is equivalent to the following finitary version:*

A theory has TP_2 if there exists a formula $\psi(\bar{x};\bar{y})$ and a natural number k such that for any natural numbers n and m we can find an array of parameters $(\bar{a}_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq m)$ satisfying the following properties:

- *$\{\psi(\bar{x};\bar{a}_{i,j}) : j \leq m\}$ is k-inconsistent for every i;*
- *$\{\psi(\bar{x};\bar{a}_{i,f(i)}) : i \leq n\}$ is consistent for every $f : \{1,\ldots,n\} \to \{1,\ldots,m\}$.*

In this paper we will prove the following.

Theorem 3.2. *Let G be a group definable in an NTP_2 theory, H be a subgroup of G and $\mathcal{G}$ be an $|H|^+$-saturated extension of G. Then the following hold:*

(1) *If H is abelian, then there exists a definable almost abelian (thus finite-by-abelian) subgroup of $\mathcal{G}$ which contains H. Furthermore, if H was normal in $\mathcal{G}$, the definable finite-by-abelian subgroup can be chosen to be normal in $\mathcal{G}$ as well.*
(2) *If H is a solvable subgroup of class n which is normal in $\mathcal{G}$, then there exists a definable normal solvable subgroup S of $\mathcal{G}$ of class at most $2n$ which almost contains H.*
(3) *If H is a nilpotent subgroup of class n, then there exists a definable nilpotent subgroup N of $\mathcal{G}$ of class at most $2n$ which almost contains H. Moreover, if H is normal in $\mathcal{G}$, the group N can be chosen to be normal in $\mathcal{G}$ as well.*

In the abelian and solvable case we follow some of the ideas already present in the proof of de Aldama. Similar to his proof and unlike the proof of Milliet in simple theories, we do not rely on a chain condition for uniformly definable subgroups, but we look to prove the result directly from the non existence of the array described in Definition 3.1. In the nilpotent case, we use additionally some properties of the almost centralizer (see Definition 3.9) needed to prove the same result in groups which satisfy the chain condition on centralizers up to finite index presented in [3].

The following is the key lemma for the abelian case and it is used as well in the nilpotent case.

Lemma 3.3. *Let G be a group with an NTP_2 theory and let H be a subgroup of G. Fix $\mathcal{G}$ an $|H|^+$-saturated extension of G and let $\phi(x,y)$ be the formula $x \in C_{\mathcal{G}}(y)$. Consider the following partial types:*

$$\pi_{Z(H)}(x) = \{\phi(x,g) : Z(H) \leq \phi(\mathcal{G},g),\ g \in \mathcal{G}\}$$
$$\pi_H(x) = \{\phi(x,g) : H \leq \phi(\mathcal{G},g),\ g \in \mathcal{G}\}.$$

Then there exists a natural numbers n such that

$$\pi_{Z(H)}(x_0) \cup \cdots \cup \pi_{Z(H)}(x_n) \cup \pi_H(y) \vdash \bigvee_{i \neq j} \phi(x_i^{-1}x_j, y).$$

Proof. Suppose that the lemma is false. Then for arbitrary large $n \in \mathbb{N}$ one can find a sequence of elements $(a_{l,0}, \ldots, a_{l,n-1}, b_l)_{l<\omega}$ in $\mathcal{G}$ such that

$$(\bar{a}_l, b_l) \models \pi_{Z(H)}(x_0) \cup \cdots \cup \pi_{Z(H)}(x_{n-1}) \cup \pi_H(y) \restriction \mathrm{dcl}(H \cup \{\bar{a}_k, b_k : k < l\})$$

and for all $0 \leq i < j < n$ we have that $a_{l,i}^{-1}a_{l,j} \notin C_{\mathcal{G}}(b_l)$. We show that:

(1) For all $i < n$ and all natural numbers k different than l, we have that $a_{l,i} \in C_{\mathcal{G}}(b_k)$;
(2) For all $i, j < n$ and all $k < l < \omega$ we have that $a_{l,i} \in C_{\mathcal{G}}(b_k^{a_{k,j}})$.

To do so, we let $k < l < \omega$ and $i, j < n$ be arbitrary and we prove that $a_{l,i} \in C_{\mathcal{G}}(b_k)$ as well as $a_{k,i} \in C_{\mathcal{G}}(b_l)$ and $a_{l,i} \in C_{\mathcal{G}}(b_k^{a_{k,j}})$.

Let z be an element of $Z(H)$. Hence H is a subgroup of $C_{\mathcal{G}}(z)$ and whence $\phi(x,z) \in \pi_H(x) \upharpoonright H$. As b_k satisfies this partial type, we obtain that

$$Z(H) \leq C_{\mathcal{G}}(b_k).$$

So $\phi(x,b_k)$ belongs to $\pi_{Z(H)}(x) \upharpoonright \{b_k\}$. Since the element $a_{l,i}$ satisfies $\pi(x)_{Z(H)} \upharpoonright H \cup \{b_k\}$, we get that $a_{l,i}$ belongs to $C_{\mathcal{G}}(b_k)$.

On the other hand, if we take $a \in H$ we have that $Z(H)$ is a a subgroup of $C_{\mathcal{G}}(a)$ and thus $\phi(x,a) \in \pi_{Z(H)}(x) \upharpoonright H$. As $a_{k,i}$ satisfy this partial type, we obtain that

$$H \leq C_{\mathcal{G}}(a_{k,i}).$$

So $\phi(x,a_{k,i}) \in \pi_H(x) \upharpoonright \{a_{k,i}\}$. As the element b_l satisfies this partial type $\pi_H(x) \upharpoonright H \cup \{a_{k,i}\}$, we get that the element $a_{k,i}$ belongs to $C_{\mathcal{G}}(b_l)$ which together with the previous paragraph yields (1).

As seen before, we have that $Z(H) \leq C_{\mathcal{G}}(b_k)$ and $H \leq C_{\mathcal{G}}(a_{k,i})$. This yields that $Z(H) \leq C_{\mathcal{G}}(b_k^{a_{k,j}})$. Hence $\phi(x, b_k^{a_{k,j}})$ belongs to $\pi_{Z(H)}(x) \upharpoonright \mathrm{dcl}\{b_k, a_{k,j}\}$. Since the element $a_{l,i}$ satisfies $\pi(x)_{Z(H)} \upharpoonright \mathrm{dcl}(H \cup \{b_k, a_{k,j}\})$, we obtain that $a_{l,i}$ belongs to $C_{\mathcal{G}}(b_k^{a_{k,j}})$ which yields (2).

Let $\psi(x;y,z)$ be the formula that defines the coset of $y \cdot C_{\mathcal{G}}(z)$. We claim that the following holds:

- $\{\psi(x;a_{l,i},b_l\} : i < n\}$ is 2-inconsistent for any $l \in \omega$;
- $\{\psi(x;a_{l,f(l)},b_l\} : l \in \omega\}$ is consistent for any function $f : \omega \to n+1$.

The first family is 2-inconsistent as every formula defines a different coset of $C_{\mathcal{G}}(b_l)$ in $\mathcal{G}$. For the second we have to show that for all natural numbers m and all tuples $(i_0, \ldots, i_m) \in n^m$ the intersection

$$a_{0,i_0} C_{\mathcal{G}}(b_0) \cap \cdots \cap a_{m,i_m} C_{\mathcal{G}}(b_m)$$

is nonempty. Using (1) and (2) and multiplying by $a_{0,i_0}^{-1} \cdot \cdots \cdot a_{m,i_m}^{-1}$ on the right, it is equivalent to $C_{\mathcal{G}}(b_0^{a_{0,i_0}}) \cap \cdots \cap C_{\mathcal{G}}(b_m^{a_{m,i_m}})$ being nonempty which is trivial true.

Compactness yields a contradiction to the fact that the group G has an NTP_2 theory and we obtain the result. $\square$

3.1. **Abelian subgroups.**

Proof of Theorem 3.2(1). Since H is abelian, it is equal to its center. So by Lemma 3.3 and compactness one can find a finite conjunction $\bigwedge_i \phi(x, g_i)$ with $\phi(x,y)$ being the formula $x \in C_{\mathcal{G}}(y)$ and g_i in some saturated extension of G, such that

$$\bigwedge_i \phi(x_0, g_i) \wedge \cdots \wedge \bigwedge_i \phi(x_n, g_i) \wedge \bigwedge_i \phi(y, g_i) \vdash \bigvee_{i \neq j} \phi(x_i^{-1} x_j, y). \quad (*)$$

Furthermore, all h in H satisfies $\bigwedge_i \phi(x, g_i)$. Hence the subgroup $\bigcap_i C_{\mathcal{G}}(g_i)$ contains H and by $(*)$, it is a bounded almost abelian group. Thus, its commutator subgroup is finite by Fact 2.10, which yields Theorem 3.2(1). Moreover, if H is normal in $\mathcal{G}$, the group $\bigcap_i (C_{\mathcal{G}}(g_i)^{\mathcal{G}})$ is a definable normal subgroup of $\mathcal{G}$ which still contains H and which is as well almost abelian, which completes the proof. $\square$

3.2. **Solvable subgroups.** To prove the solvable case of Theorem 3.2 we need the following.

Definition 3.4. *A group G is* almost solvable *if there exists a normal* almost series *of finite length, i. e. a finite sequence of normal subgroups*

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

of G such that G_{i+1}/G_i is an almost abelian group for all $i \in n$. The least such natural number $n \in \omega$ is called the almost solvable class *of G.*

Definition 3.5. *Let G be a group and S be a definable almost solvable subgroup of class n. We say that S* admits a definable almost series *if there exists a family of definable normal subgroups $\{S_i : i \leq n\}$ of S such that S_0 is the trivial group, S_n is equal to S and S_{i+1}/S_i is almost abelian.*

The proof of Corollary 4.10 in [4] provides the following fact (although it is done in the context of a simple theory, the proof is exactly the same in our context). It can also been found as Lemma 3.22 in [3].

Fact 3.6. *Let G be a definable almost solvable subgroup of class n which admits a definable almost series. Then G has a definable subgroup of finite index which is solvable of class at most $2n$.*

So we only need to concentrate on building a definable almost series.

Proposition 3.7. *Let G be a group definable in an NTP$_2$ theory and H be a solvable subgroup of G of class n. Suppose that there is a $|H|^+$-saturated extension $\mathcal{G}$ of G which normalizes H. Then there exists a definable normal almost solvable subgroup S of $\mathcal{G}$ of class n containing H. Additionally, S admits a definable almost series such that all of its members are normal in $\mathcal{G}$.*

Proof. We prove this by induction on the derived length n of H. If n is equal to 1 this is a consequence of the abelian case, Theorem 3.2(1). So let $n > 1$, and consider the abelian subgroup $H^{(n-1)}$ of H. It is a characteristic subgroup of H and hence, as H is normal in $\mathcal{G}$, it is normal in $\mathcal{G}$ as well. So again by the abelian case, there exists a definable almost abelian normal subgroup A of $\mathcal{G}$ which contains $H^{(n-1)}$. Replacing $\mathcal{G}$ by $\mathcal{G}/A$, we have that the derived length of HA/A is at most $n-1$ and we may apply the induction hypothesis which finishes the proof. $\square$

Proof of Theorem 3.2(2). Applying Proposition 3.7 to H seen as a normal subgroup of $\mathcal{G}$ gives us a definable almost solvable subgroup K of $\mathcal{G}$ of class n containing H and which admits a definable almost series. By Fact 3.6, the group K has a definable subgroup S of finite index which is solvable of class at most $2n$. □

3.3. **Nilpotent subgroups.** The following follows from Lemma 3.3

Lemma 3.8. *Let G be a group definable in an NTP$_2$ theory, let H be a subgroup of G and suppose that G is $|H|^+$ saturated. Then one can find definable subgroups A and K and a natural number m such that*

- *the cardinality of the conjugacy class k^A for all elements k in K is bounded by m;*
- *A is almost abelian and contains $Z(H)$;*
- *K contains H and A.*

If H is additionally normal in G, one can choose A and K to be normal in G as well.

Proof. By Lemma 3.3 we can find $\phi_{Z(H)}$ and ϕ_H which are conjunctions of formulas from $\pi_{Z(H)}(x)$ and $\pi_H(x)$ (defined as in Lemma 3.3) respectively and a natural number m such that:

$$\phi_{Z(H)}(x_0) \wedge \cdots \wedge \phi_{Z(H)}(x_m) \wedge \phi_H(y) \vdash \bigvee_{i \neq j} x_i^{-1} x_j \in C_{\mathcal{G}}(y).$$

Note that these formulas define intersections of centralizers and are therefore subgroups of G. Letting A be equal to $\phi_{Z(H)}(G) \cap \phi_H(G)$ and K be equal to $\phi_H(G)$ we have the announced properties.

If H is normal in G, we have that $Z(H)$ is also normal in G and we can replace A and K by $\cap_{g \in G} A^g$ and $\cap_{g \in G} K^g$ which are normal definable subgroups of G and still satisfy the given properties. □

To prove the existence of "definable envelopes" of nilpotent subgroups of a group definable in an NTP$_2$ theory we need to define the almost-centralizer.

Definition 3.9. *Let G be a group an H be a definable subgroup of G. We define the almost-centralizer $\widetilde{C}_G(H)$ of H in G to be*

$$\widetilde{C}_G(H) := \{g \in G \mid [H : C_H(g)] < \infty\}.$$

We will need the following results, which are Corollary 2.11 and Proposition 3.27 in [3].

Fact 3.10. ***(Symmetry)*** *Let G be a group and let H and K be two definable subgroups of G. So*

$$H \lesssim \widetilde{C}_G(K) \text{ if and only if } K \lesssim \widetilde{C}_G(H).$$

Fact 3.11. *Let G be a group and let H and K be two definable subgroups of G such that H is normalized by K. Suppose that H is contained in $\widetilde{C}_G(K)$ and K is contained in $\widetilde{C}_G(H)$. Then $[H, K]$ is finite.*

We will also need a theorem which is the definable version of a result proven by Schlichting in [9] and which can be found in [11] as Theorem 4.2.4. It deals with families of uniformly commensurable subgroups, a notion we now introduce:

Definition 3.12. *A family $\mathcal{F}$ of subgroups is* uniformly commensurable *if there exists a natural number d such that for each pair of groups H and K from $\mathcal{F}$ the index of their intersection is smaller than d in both H and K.*

Fact 3.13. *Let G be a group and $\mathcal{H}$ be a family of definable uniformly commensurable subgroups. Then there exists a definable subgroup N of G which is commensurable which all elements of $\mathcal{H}$ and which is invariant under any automorphisms of G which stabilizes $\mathcal{H}$ setwise.*

Proof of Theorem 3.2(3). Note that if H is finite the result holds trivially. So we may assume that H is infinite and suppose as well that G is already $|H|^+$-saturated. Thus for the second part of the theorem, we may assume that H is normal in G.

We will prove by induction on the nilpotency class n of H that there exists a definable nilpotent subgroup N of G of class at most $2n$ and a sequence of subgroups:
$$\{1\} = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{2n} = N$$
such that $H \lesssim N$ and for all $0 \leq i < 2n$, we have that

- N_i is definable and normal in N;
- $[N_{i+1}, N] \leq N_i$.

If H was supposed to be normal in G, we will have each N_i to be normal in G as well.

Let n be equal to 1. Then H is abelian, and by Theorem 3.2(1) there exists a definable almost abelian subgroup A of G which contains H. Note that the centralizer of any element of A has finite index in A. As $[A, A]$ is finite by Fact 2.10, letting N be equal to $C_A([A, A])$ and $N_1 = Z([A, A])$ gives the desired groups. If H was assumed to be normal in G, we may choose A to be normal in G as well. Since $C_A([A, A])$ and $Z([A, A])$ are characteristic subgroups of A, they will be also normal in G which provides the second part of the theorem.

Assume now n is strictly greater than 1. Assume that for any nilpotent subgroup of a definable group in an NTP$_2$ theory of class less or equal to n, one can find a sequence as described above. The strategy is to find a definable subgroup N^* of G such that N^* almost contains H and $Z_2(N^*)$ contains $N^* \cap Z(H)$. We will then prove that in this case $(H \cap N^*)\, Z_2(N^*)/Z_2(N^*)$ has nilpotency class strictly smaller than H so that we may apply the induction hypothesis to find a definable nilpotent subgroup $N_{2n}/Z_2(N^*)$ of $G/Z_2(N^*)$ which almost contains $N^*/Z_2(N^*)$ and therefore $H/Z_2(N^*)$. Taking the pullback to N^* together with its first and second center yields the desired properties.

We first show the following:

Claim. *There are definable subgroups A and K of G such that:*

- *A is a normal subgroup of K;*
- *$Z(H) \lesssim A$ and $H \leq K$;*
- *$K \leq \widetilde{C}_K(A)$;*
- *$A \leq \widetilde{C}_K(K)$;*
- *$[A, K]$ is finite and contained in $\widetilde{C}_G(K)$.*

Proof. First, by Lemma 3.8 we can find definable subgroups A_0 and K of G such that

(1) the cardinality of the conjugacy class k^{A_0} for all elements k in K is bounded by some natural number m;
(2) A_0 is almost abelian and contains $Z(H)$;
(3) K contains H and A_0.

The next step to prove the claim is to replace A_0 by a commensurable definable subgroup which is additionally normal in K.

By (1) we can deduce that K is contained in $\widetilde{C}_G(A_0)$ and therefore

$$\mathcal{F} = \{{A_0}^k : k \in K\}$$

is a uniformly definable and uniformly commensurable family of subgroups of K. By Schlichting (Fact 3.13) one can find a definable subgroup A_1 of K which is commensurable with all groups in $\mathcal{F}$, in particular with A_0, and which is stabilized by all automorphisms which stabilize the family setwise, and thus is normal in K.

As A_1 is commensurable with A_0, we have that $K \leq \widetilde{C}_K(A_1)$. By symmetry of the almost centralizer, we obtain that A_1 is almost contained in $\widetilde{C}_K(K)$, but A_1 need not be a subgroup of $\widetilde{C}_K(K)$. Let $A = A_1 \cap \widetilde{C}_K(K)$; this is still a normal subgroup of K and has finite index in A_1, which will imply all of the group theoretic properties in the statement. However, the almost centralizer of a definable group is not necessary definable, so it is left to show that this intersection is indeed definable.

Since A has finite index in A_1, the definable subgroup A_1 is a finite union of distinct cosets of A, say $A_1 = \bigcup_{i=1}^{k} a_i A$ for some $a_i \in A_1$. Furthermore, we have that A is the union of the definable sets

$$A_d := \phi_d(x) = \{x \in A_1 : [K : C_K(x)] < d\}.$$

But then we have that

$$A_1 = \bigcup_{i=1}^{k} \bigcup_{d \in \mathbb{N}} a_i A_d$$

so by compactness and saturation of G this is equal to a finite subunion. Additionally, as $\{A_d\}_{d\in\omega}$ was a chain of subsets of A each contained in the next we have that

$$A_1 = \bigcup_{i=1}^{k} a_i A_d$$

for some fixed d. Hence A is equal to A_d and whence it is a normal definable subgroup of K. Moreover, the group A is commensurable with A_0, so it almost contains $Z(H)$ and K is still in $\widetilde{C}_G(A)$. Additionally, A is contained in $\widetilde{C}_G(K)$ and normal in K, so Fact 3.11 implies that the group $[A, K]$ is finite and contained in both A and in $\widetilde{C}_K(K)$. $\square$

Let A and K be as in the claim, so the index $[Z(H) : A \cap Z(H)]$ is finite. Take a set $H_0 := \{h_0, \ldots, h_n\}$ of representatives of each classes of $A \cap Z(H)$ in $Z(H)$, so that $Z(H) = h_0 (A \cap Z(H)) \cup h_1 (A \cap Z(H)) \cup \cdots \cup h_n (A \cap Z(H))$.

Let $K' := C_K(h_0, \ldots, h_n)$ and $A' := A \cap K'$.

Claim. *The following conditions hold:*

- $[A', K']$ *is finite and contained in* $\widetilde{C}_G(K')$.
- $H \leq K'$.
- $Z(H) \cap A = Z(H) \cap A'$, *so that* $Z(H) \lesssim A'$.

Proof. Since $K' \leq K$ and $A' \leq A$, we have that $Z(H) \cap A' \subseteq Z(H) \cap A$ and $[A', K'] \leq [A, K]$. Since $[A, K]$ is finite and contained in $\widetilde{C}_G(K)$, so is $[A', K']$. Furthermore, we have that $\widetilde{C}_G(K)$ is a subgroup of $\widetilde{C}_G(K')$. This yields the first item of the claim.

All of the h_i's in H_0 belong to $Z(H)$ and H is a subgroup of K, so $H \leq K' = C_K(h_0, \ldots, h_n)$.

Finally, let h be an element of $Z(H) \cap A$. We have that h belongs as well to K' and hence to A'. This completes the proof of the claim. $\square$

Notice that in particular $Z(H) \cap A \leq A'$.

We can now define N^* as we mentioned at the beginning of the proof.

Let X be equal to $[A', K']$. Then we define:

$$N^* := C_{K'}(X).$$

Claim. *The following conditions hold:*

(1) N^* is a subgroup of K' of finite index, and thus $H \cap N^*$ has finite index in H.
(2) $X \cap N^* \leq Z(N^*)$.
(3) $Z(H) \cap N^* \leq Z_2(N^*)$.

Proof. Since X is contained in $\widetilde{C}_G(K')$, the definition yields that $C_{K'}(x)$ has finite index in K'. As X is additionally finite, we obtain that N^* has finite index in K'. Since H is a subgroup of K', we have as well that $H \cap N^*$ has finite index in H as well, which proves (1).

As N^* is equal to $C_{K'}(X)$, we obtain immediately that $X \cap N^*$ is contained in $Z(N^*)$.

To prove (3), it is enough to show that for given z in $Z(H) \cap N^*$ and $n \in N^*$, the commutator $[z, n]$ belongs to $Z(N^*)$. This will imply that $[Z(H) \cap N^*, N^*] \leq Z(N^*)$ which yields that $Z(H) \cap N^*$ is contained in $Z_2(N^*)$.

As

$$Z(H) = \bigcup_{h_i \in H_0} h_i \left(A' \cap Z\left(H\right)\right),$$

we can write z as a product of an element $h_i \in H_0$ and $a \in A'$. Thus

$$[z, n] = [h_i \cdot a, n] = [h_i, n]^a \cdot [a, n]$$

As n belongs to N^* which is a subgroup of $K' = C_K(H_0)$, the first factor is trivial and we obtain that:

$$[z, n] = [a, n] \in [A', K'] \leq X.$$

Moreover, as z and n both belong to N^*, their commutator does as well. Thus we obtain finally that $[z, n]$ is an element of $X \cap N^*$ which is a subgroup of $Z(N^*)$ as shown above. This completes the proof of the claim. $\square$

We are finally ready to prove the theorem, using the induction hypothesis.

By the previous claim, we have that $Z(H) \cap N^* \leq Z_2(N^*)$. Hence

$$\left(H \cap N^*\right)/Z_2(N^*) \cap \left(H \cap N^*\right) \cong \left(H \cap N^*\right) Z_2(N^*)/Z_2(N^*)$$

is a quotient of $\left(H \cap N^*\right) / \left(Z\left(H\right) \cap N^*\right)$. We obtain that the nilpotency class of $\left(H \cap N^*\right) Z_2(N^*)/Z_2(N^*)$ is at most the nilpotency class of $H/Z(H)$ which is strictly smaller than the one of H. Furthermore, it is contained in the group $N^*/Z_2(N^*)$ which is definable in an NTP_2 theory.

By induction hypothesis, we can find a sequence of subgroups of $N^*/Z_2(N^*)$

$$Z_2(N^*)/Z_2(N^*) \leq N_3/Z_2(N^*) \leq \cdots \leq N_{2n}/Z_2(N^*)$$

such that

$$\left(H \cap N^*\right) Z_2(N^*)/Z_2(N^*) \lesssim N_{2n}/Z_2(N^*)$$

and for all $2 \le i \le 2n$ we have that

- $N_i/Z_2(N^*)$ is definable and normal in $N_{2n}/Z_2(N^*)$;
- $[N_{i+1}, N_{2n}] \le N_i$.

As N_{2n} is a subgroup of N^* we have that $Z(N^*) \cap N_{2n} \le Z(N_{2n})$ and $[Z_2(N^*), N_{2n}] \le Z(N^*)$. Note that the group $H \cap N^*$ is as well almost contained in N_{2n}. As $H \cap N^*$ and H are commensurable, the same holds for H. So

$$\{1\} = N_0 \le Z(N_{2n}) \le Z_2(N_{2n}) \le N_3 \cdots \le N_{2n}$$

is an ascending central series of N_{2n} with the desired properties.

For the "moreover" part of Theorem 3.2(3), if H is normal in G, then the groups:

$$L := K^G \cap C_G(h_0^G, \dots, h_n^G) \quad \text{and} \quad B = A^G \cap L$$

are normal subgroups of G and we have as well that:

- $H \le L$ and $Z(H) \lesssim B$, so in particular $Z(H) \cap A \le B$;
- $[B, L]$ is finite and contained in $\widetilde{C}_G(L)$.

Doing the same construction to find N^* using L and B instead of K' and A', we have additionally that N^* is normal in G, which implies that both $Z(N^*)$ and $Z_2(N^*)$ are also normal subgroups of G. The rest of the proof is exactly the same. □

Corollary 3.14. *Let G be a group with an NTP$_2$ theory and let H be a nilpotent subgroup of G of class n. Suppose there exists an $|H|^+$-saturated extension $\mathcal{G}$ of G which normalizes H, then there is a definable nilpotent normal subgroup N of $\mathcal{G}$ of class at most $3n$ which contains H.*

Proof. By Theorem 3.2(3), there is a definable nilpotent subgroup N_0 of class at most $2n$ which almost contains H and which is normal in $\mathcal{G}$. Thus, the group NH is a definable normal nilpotent subgroup of $\mathcal{G}$ of nilpotency class at most $3n$. □

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